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Gravity on an extended brane in six-dimensional warped flux compactifications

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We study linearized gravity in a six-dimensional Einstein-Maxwell model of warped braneworlds, where the extra dimensions are compactified by a magnetic flux. It is difficult to construct a strict codimension two braneworld with matter sources other than pure tension. To overcome this problem we replace the codimension two defect by an extended brane, with one spatial dimension compactified on a Kaluza-Klein circle. Our background is composed of a warped, axisymmetric bulk and one or two branes. We find that weak gravity sourced by arbitrary matter on the brane(s) is described by a four-dimensional scalar-tensor theory. We show, however, that the scalar mode is suppressed at long distances and hence four-dimensional Einstein gravity is reproduced on the brane.

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I. INTRODUCTION

Higher dimensional theories have been attracting much attention for many years. In the traditional Kaluza-Klein theories the compactification scale must be microscopic to guarantee effectively four-dimensional (4D) spacetime. In contrast to the Kaluza-Klein picture, recently proposed braneworld scenarios [1] allow for the presence of large (or even infinite) extra dimensions owing to localization of matter fields on the brane. Among various braneworld models, those with two extra dimensions have received particular interests from a phenomenological point of view. A naive reduction of six-dimensional (6D) gravity leads to $M_{\rm Pl}^2 = M^4\mathcal{V}$ [2], where $M_{\rm Pl}$ is the observed 4D Planck mass and M is the fundamental scale of gravity, and $\mathcal{V}$ is the volume of extra dimensions. For $M \sim 1$ TeV we roughly have $\mathcal{V}^{1/2} \sim 1$ mm, which eliminates the hierarchy problem in the usual sense and at the same time offers us the possibility to detect extra dimensions by table-top experiments on gravity.

In the present paper we focus on the 6D brane model in which the internal two-dimensional space is compactified by a magnetic flux. (See, e.g., Ref. [3] and references therein for recent progresses in 6D braneworld models.) The basic motivation for considering such models is that extra dimensions, branes, and fluxes are important ingredients in string theory. Braneworld settings with flux compactification provide us natural toy models unifying aspects of string theory and cosmology, though such scenarios have not been explored much. Moreover, it has been suggested that the football (or rugby-ball) shaped codimension two braneworlds may help to resolve the cosmological constant problem by a self-tuning mechanism [4]. (The mechanism has been criticized for several reasons [5].)

The behavior of gravity in 6D braneworlds should be investigated carefully. In order for the brane scenario to be "realistic", it is necessary to reproduce 4D Einstein gravity on the brane at least on scales much larger than the compactification radius. Properties of gravity in a codimension two braneworld with flux compactification has been discussed so far in Ref. [6]. However, a codimension two brane suffers from the problem of the localization of matter. Namely, it is difficult to put energy-momentum tensor different from pure tension on the brane in Einstein gravity [7, 8]. Actually it has been demonstrated that massless relativistic particles can also be accommodated on a conical brane; the true problem here lies in the fact that the gravitational radius of a mass m is enhanced on the brane and hence gravity becomes nonlinear at distances much greater than a naive estimate [9][1]. In any case the description of gravity on a conical defect is involved, which significantly restricts the analysis of gravitational aspects of 6D braneworlds. One way to evade this generic problem is introducing codimension one brane with one extra dimension compactified on a circle (which we call an *extended brane*) instead of the original codimension two defect [10]. (See Refs. [8, 11, 12, 13, 14, 15, 16] for other ways out.) Since the model has a Kaluza-Klein spatial direction in addition to one large extra dimension, this construction is thought of as a "hybrid" braneworld scenario [17]. (Very recently this approach has been employed to regularize a conical singularity in a different context [18].)

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Along this line, the authors of Ref. [10] were the first to study the behavior of gravity in the presence of matter sources on the brane. In their model the internal space is football shaped (i.e., not warped) and codimension two branes at the poles are replaced by the extended branes wrapped around the axis of the football. The perturbation analysis shows that standard 4D gravity is reproduced on the extended brane [10]. The system is stable under the most generic axisymmetric perturbations [19]. In Ref. [20] the same regularization technique is used for more general brane models with *warped* internal space both in Einstein-Maxwell theory [6] and in 6D supergravity [21, 22]. In this paper we extend the work of [10] and investigate linearized gravity sourced by arbitrary matter on extended branes in the Einstein-Maxwell model of the warped braneworld [20]. As in [10], we basically follow the well-developed analysis of perturbations [23] in the Randall-Sundrum braneworld [24]. This model will serve as a simple playground to study gravitational aspects of braneworld scenarios in the context of string flux compactifications.

The paper is organized as follows. In Sec II we describe our background model. Then we derive perturbation equations and boundary conditions in Sec. III. In Sec. IV we discuss the behavior of linearized gravity on the brane. Finally we draw our conclusions in Sec. V.

II. 6D WARPED FLUX COMPACTIFICATION AND EXTENDED BRANES

A. Bulk solution with extended branes

The 6D background model we will discuss was originally used to compactify an internal two-dimensional space in Einstein-Maxwell theory [25]. We consider the system in the braneworld context, and basically follow the construction of [10, 20]. The bulk has a structure of Minkowski spacetime times a two-dimensional compact space, and consists of $M+1$ regions ($\mathcal{M}_I$) separated by M extended branes (Σ_i). Our action is

$$S=\sum_I S_I+\sum_i S_i, \tag{2.1}$$

where the I-th bulk and i-th brane actions are given, respectively, by

$$S_I = \int_{\mathcal{M}_I} d^6x\sqrt{-g}\left[\frac{M^4}{2}\left(\mathcal{R}-\frac{1}{L_I^2}\right)-\frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN}\right], \tag{2.2}$$

$$S_i = -\int_{\Sigma_i} d^5x\sqrt{-q}\left[\lambda_i+\frac{v_i^2}{2}q^{\hat\mu\hat\nu}\left(\partial_{\hat\mu}\sigma_i-\mathrm{e}\mathcal{A}_{\hat\mu}\right)\left(\partial_{\hat\nu}\sigma_i-\mathrm{e}\mathcal{A}_{\hat\nu}\right)\right]. \tag{2.3}$$

Here the capital Latin indices $\{M,N,\cdots\}$ and the Greek indices with hat $\{\hat\mu,\hat\nu,\cdots\}$ are used for tensors defined in the 6D bulk and on the 5D brane(s), respectively. We have the $U(1)$ gauge field $\mathcal{A}_M$ with field strength $\mathcal{F}_{MN}=\partial_M\mathcal{A}_N-\partial_N\mathcal{A}_M$ in the bulk, which is coupled to the brane scalar fields σ_i. On each brane σ_i arises as a Goldstone mode of a Higgs field, and the absolute value of the vacuum expectation value of the Higgs field is given by $v_i/\sqrt{2}$. We also include tension λ_i in the brane actions[2]. In this model the cosmological constant $1/(2L_I^2)$ is different at different regions in the bulk.

The background solution is described by the metric

$$g_{MN}dx^M dx^N=a^2(w)\eta_{\mu\nu}dx^\mu dx^\nu+L_I^2\left[\frac{dw^2}{f(w)}+\beta_I^2 f(w)d\phi^2\right], \tag{2.4}$$

where

$$f\left(w\right)=\frac{1}{5(1-\alpha)^2}\left[-a^2(w)+\frac{1-\alpha^8}{1-\alpha^3}\frac{1}{a^3(w)}-\frac{\alpha^3(1-\alpha^5)}{1-\alpha^3}\frac{1}{a^6(w)}\right], \tag{2.5}$$

and the field strength

$$\mathcal{F}_{w\phi}=\frac{\mu M^2\beta_I L_I}{a^4},\quad \mu:=\sqrt{\frac{3\alpha^3(1-\alpha^5)}{5(1-\alpha^3)}}. \tag{2.6}$$

[2] The brane action of the form (2.3) may not be unique but rather is the simplest one. See Ref. [21] for brane actions.

The warp factor is given by

$$a(w) = \frac{1}{2}\left[(1-\alpha)w + 1 + \alpha\right], \tag{2.7}$$

and α $(0 < \alpha \leq 1)$ characterizes warping of the bulk. The function $f(w)$ has two positive roots $w = 1$ $(a = 1)$ and $w = -1$ $(a = \alpha)$, and hence we will consider the space $-1 \leq w \leq 1$ $(\alpha \leq a \leq 1)$. The original construction of the braneworld model of this type is given by Mukohyama et al. in Ref. [6], in which codimension two defects are considered. The stability analysis against classical perturbations has been performed in their subsequent papers [26, 27]. In Appendix A we replicate the detailed derivation of the bulk solution [6, 20].

We put brane(s) at $w = \bar{w}_i$. Then the continuity of the $(\phi\phi)$ component of the induced metric implies that $\beta_I L_I$ is continuous across the brane. Therefore we write

$$\ell := \beta_I L_I. \tag{2.8}$$

The points $w = \pm 1$ can be regarded as poles. We assume that ϕ has period 2π. In order to avoid conical singularities at the two poles, we impose

$$\beta_+ = \frac{20(1-\alpha)(1-\alpha^3)}{5 - 8\alpha^3 + 3\alpha^8}, \qquad \beta_- = \frac{20(1-\alpha^{-1})(1-\alpha^{-3})}{5 - 8\alpha^{-3} + 3\alpha^{-8}}, \tag{2.9}$$

where $\beta_\pm$ denotes the value of β_I in the region that contains the pole $w = \pm 1$. It is easy to see that $\beta_+ \geq 1 \geq \beta_- > 0$. Now we check that the above bulk solution coincides with the model of [10] in the unwarped limit $\alpha \to 1$. Indeed, we have $f \to 1 - w^2$, $\mathcal{F}_{w\phi} \to M^2 \beta_I L_I$, and $\beta_\pm \to 1$ as $\alpha \to 1$. With the coordinate transformation $w = \sin\theta$ we can reproduce the bulk solution of [10]. In this unwarped limit the background has a Z_2 symmetry across the equator $(w = 0)$ before introducing branes.

Let us look at the quantization condition for the flux. Since $\mathcal{F}_{w\phi} = \mathcal{A}'_\phi$, where the prime stands for a derivative with respect to w, we have

$$\mathcal{A}_\phi = \mathcal{A}_\phi^{(N)} := -\frac{2\mu\ell M^2}{3(1-\alpha)}\left(\frac{1}{a^3} - 1\right) \quad \text{for} \;\; w > w_e - \varepsilon, \tag{2.10}$$

$$\mathcal{A}_\phi = \mathcal{A}_\phi^{(S)} := -\frac{2\mu\ell M^2}{3(1-\alpha)}\left(\frac{1}{a^3} - \frac{1}{\alpha^3}\right) \quad \text{for} \;\; w < w_e + \varepsilon, \tag{2.11}$$

where w_e is arbitrary and $\varepsilon > 0$. The integration constants here are chosen so that $\mathcal{A}_\phi = 0$ at the poles. We see that in the overlapping region $w_e - \varepsilon < w < w_e + \varepsilon$, $\mathcal{A}_\phi$ is doubly defined. This means that $\mathcal{A}_\phi^{(N)}$ and $\mathcal{A}_\phi^{(S)}$ are related via the following gauge transformation:

$$\mathcal{A}_\phi^{(S)} = \mathcal{A}_\phi^{(N)} + \partial_\phi\left(\mu_* \ell M^2 \,\phi\right), \qquad \mu_* := \sqrt{\frac{(1-\alpha^3)(1-\alpha^5)}{15\alpha^3(1-\alpha)^2}}. \tag{2.12}$$

Noting that σ_i is the phase of the brane Higgs field with charge e under the $U(1)$ symmetry, we arrive at the following quantization condition:

$$2\mathrm{e}\,\mu_* \ell M^2 = N, \quad N = 0, 1, 2, \cdots. \tag{2.13}$$

The junction equations at a brane are given as follows. They are derived from (i) the continuity of the induced metric, (ii) the Israel conditions, and (iii) the junction conditions for the Maxwell field. The first one was already used to relate β_I with L_I in Eq. (2.8). The Israel conditions are written as

$$[[K_{\hat\mu\hat\nu} - q_{\hat\mu\hat\nu} K]]_{\bar{w}_i} = -\frac{S^{(i)}_{\hat\mu\hat\nu}}{M^4}, \tag{2.14}$$

where $S_{\hat\mu\hat\nu}$ is the energy-momentum tensor on the brane and $[[F]]_{\bar{w}_i} := \lim_{\epsilon\to 0}\left(F|_{\bar{w}_i+\epsilon} - F|_{\bar{w}_i-\epsilon}\right)$. It is straightforward to compute the extrinsic curvature. The energy-momentum tensor on the brane is given by

$$S^{(i)}_{\mu\nu} = -\left[\lambda_i + \frac{v_i^2}{2} q^{\phi\phi}\left(\partial_\phi\sigma_i - \mathrm{e}\mathcal{A}_\phi\right)^2\right] q_{\mu\nu}\Big|_{\bar{w}_i}, \tag{2.15}$$

$$S^{(i)}_{\phi\phi} = -\left[\lambda_i - \frac{v_i^2}{2} q^{\phi\phi}\left(\partial_\phi\sigma_i - \mathrm{e}\mathcal{A}_\phi\right)^2\right] q_{\phi\phi}\Big|_{\bar{w}_i}, \tag{2.16}$$

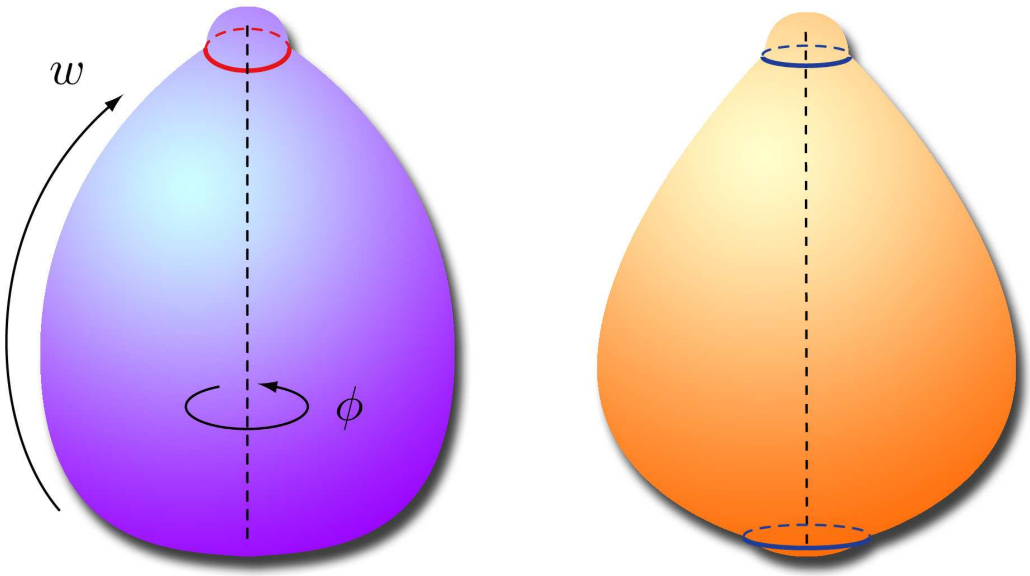

FIG. 1: Single brane and two-brane models.

Then the Israel conditions are rearranged to give

$$[[\beta_I]]_{\bar{w}_i} \cdot \frac{f^{1/2}}{\ell}\left(\frac{f'}{2f}-\frac{a'}{a}\right)\bigg|_{\bar{w}_i} = -\frac{v_i^2}{M^4} q^{\phi\phi}\left(\partial_\phi \sigma_i - \mathrm{e}\mathcal{A}_\phi\right)^2\bigg|_{\bar{w}_i}, \tag{2.17}$$

$$[[\beta_I]]_{\bar{w}_i} \cdot \frac{f^{1/2}}{\ell}\left(\frac{f'}{4f}+\frac{7}{2}\frac{a'}{a}\right)\bigg|_{\bar{w}_i} = -\frac{\lambda_i}{M^4}. \tag{2.18}$$

The junction conditions for the Maxwell field can be written as

$$\left[\left[n^N \mathcal{F}_{NM}\right]\right]_{\bar{w}_i} = -\,\mathrm{e}v_i^2\left(\partial_M \sigma_i - \mathrm{e}\mathcal{A}_M\right)\big|_{\bar{w}_i}, \tag{2.19}$$

where n^N is the unit normal to the brane. For the background, only the ϕ component is relevant:

$$[[\beta_I]]_{\bar{w}_i} \cdot \frac{\mu M^2 f^{1/2}}{a^4}\bigg|_{\bar{w}_i} = -\,\mathrm{e}v_i^2\left(\partial_\phi \sigma_i - \mathrm{e}\mathcal{A}_\phi\right)\big|_{\bar{w}_i}. \tag{2.20}$$

From Eqs. (2.17) and (2.20) we obtain

$$[[\beta_I]]_{\bar{w}_i} = -\,\ell\left(\frac{\mathrm{e}v_i}{\mu}\right)^2 \cdot a^8 f^{1/2}\left(\frac{f'}{2f}-\frac{a'}{a}\right)\bigg|_{\bar{w}_i}. \tag{2.21}$$

Since $f'/(2f) - a'/a = 0$ has a root w_c $(-1 < w_c \leq 0)$, we see that $[[\beta_I]]_{\bar{w}_i} \geq 0$ for $\bar{w}_i \geq w_c$ and $[[\beta_I]]_{\bar{w}_i} < 0$ for $\bar{w}_i < w_c$.

The equation of motion for σ_i implies $\partial^\phi\left(\partial_\phi \sigma_i - \mathrm{e}\mathcal{A}_\phi\right) = 0$, leading to $\sigma_i = n_i \phi$, where n_i is an integer. Using Eqs. (2.17) and (2.20) we obtain [20]

$$n_i = -\frac{N}{1-\alpha^3}\left[\frac{5(1-\alpha^8)}{8(1-\alpha^5)} - \alpha^3 c_i\right], \tag{2.22}$$

where $c_i = 1$ for $\bar{w}_i > w_e$ and $c_i = \alpha^{-3}$ for $\bar{w}_i < w_e$ [see Eqs. (2.10) and (2.11)]. In the limit $\alpha \to 1$, we can see that $n_i = -N/2$ for $\bar{w}_i > w_e$ and $n_i = +N/2$ for $\bar{w}_i < w_e$ [10].

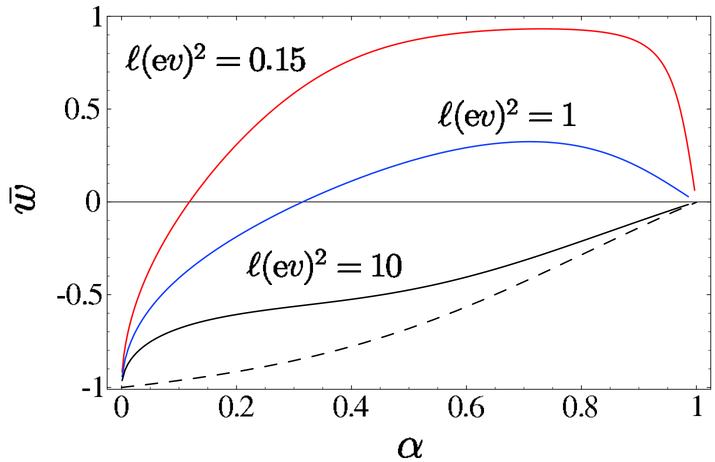

FIG. 2: The position of the brane in the single brane model. The dashed line shows the value w_c and hence one cannot put a brane below this line.

B. Single brane model

Let us consider a model with a single brane at $w = \bar{w}$. The two regions separated by the brane have the curvature scales $L_+ = \ell/\beta_+$ and $L_- = \ell/\beta_-$, respectively. The configuration of the model is described in the left side of Fig. 1. For given parameters α and $\ell(\mathrm{e}v)^2$, Eq. (2.21) determines the position of the brane $\bar{w}$, as is shown in Fig. 2. Note that one cannot put the brane in the region $w < w_c$ because of the condition $\beta_+ \geq \beta_-$.

C. Two-brane model

We can also consider a two-brane model, which is nothing but the situation considered in Ref. [20]. In this case, the bulk is divided into three parts by two branes at $w = \bar{w}_+$ and $w = \bar{w}_-$. A schematic picture is shown in the right side of Fig. 1. The three bulk regions are characterized by the curvature scales $L_+ = \ell/\beta_+$, $L_0 := \ell/\beta_0$, and $L_- = \ell/\beta_-$, respectively, from north to south. If $w_c > \bar{w}_+ > \bar{w}_-$, it follows from Eq. (2.21) that $\beta_+ < \beta_0 < \beta_-$. This is incompatible with the fact $\beta_+ \geq \beta_-$, and so we impose $\bar{w}_+ \geq w_c$. Consequently, we have $\beta_+ \geq \beta_0$. However, both $\bar{w}_- \geq w_c$ ($\beta_0 \geq \beta_-$) and $\bar{w}_- < w_c$ ($\beta_0 < \beta_-$) are possible.

The background configuration of [10] can be recovered by taking the unwarped limit ($\alpha \to 1$) and assuming at the same time that $v_+ = v_-$. As is mentioned in the above, the unwarped bulk automatically has the Z_2-symmetry across the equator at $w = 0$. The second condition requires that the branes are located at Z_2-symmetric positions, i.e., $\bar{w}_- = -\bar{w}_+$. Furthermore, in Ref. [10] the Z_2-symmetry is imposed *also for linear perturbations*. We stress that the general warped background ($\alpha \neq 1$) has no such symmetry. Our perturbation analysis in the rest of the paper does include the unwarped model as the limiting case, but we will not restrict ourselves to the Z_2-symmetric brane configuration nor perturbations in the limit $\alpha \to 1$. We will comment on this point in more detail in Sec. IV B.

III. LINEAR PERTURBATIONS

We now consider linear perturbations from the background solution described in the previous section. We write the perturbed metric in an arbitrary gauge as

$$\begin{aligned}(g_{MN} + \delta g_{MN})\, dx^M dx^N \;=\;& a^2 \left[(1+2\Psi)\eta_{\mu\nu} + 2E_{,\mu\nu} + E_{\mu,\nu} + E_{\nu,\mu} + h_{\mu\nu}\right] dx^\mu dx^\nu \\ & +2\left(B_{w,\mu} + B_{w\mu}\right) dw dx^\mu + 2\left(B_{,\mu} + B_\mu\right) d\phi dx^\mu \\ & +L_I^2\left[(1+2W)\frac{dw^2}{f} + 2C\beta_I^2 f dw d\phi + (1+2\Phi)\beta_I^2 f d\phi^2\right], \qquad (3.1)\end{aligned}$$

where the perturbations are split into scalar, vector, and tensor modes under the Lorentz group in the external spacetime. The vector modes E_μ, $B_{w\mu}$, and B_μ are transverse: $E_\mu{}^{,\mu} = 0, \cdots$. The tensor mode $h_{\mu\nu}$ is transverse and traceless: $h_{\mu\nu}{}^{,\nu} = 0 = h_\mu{}^\mu$. Similarly, we can write the perturbations of the gauge field $\delta\mathcal{A}_M$ as

$$\delta\mathcal{A}_\mu = \partial_\mu A + \hat{A}_\mu, \quad \delta\mathcal{A}_w = A_w, \quad \delta\mathcal{A}_\phi = A_\phi, \tag{3.2}$$

where $\hat{A}_\mu$ is a vector mode and the other three are scalar modes. The scalar part of the perturbed field strength $\delta\mathcal{F}_{MN}$ is given by

$$\delta\mathcal{F}_{w\phi} = A'_\phi, \quad \delta\mathcal{F}_{\mu\phi} = A_{\phi,\mu}, \quad \delta\mathcal{F}_{\mu w} = F_{w,\mu}, \tag{3.3}$$

where $F_w := A_w - A'$. The vector part of the perturbed field strength is

$$\delta\hat{\mathcal{F}}_{w\mu} = \hat{A}'_\mu, \quad \delta\hat{\mathcal{F}}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu. \tag{3.4}$$

We shall discuss tensor and scalar perturbations in the following subsections. We show in Appendix C that there are no vector-type perturbations of importance.

A. Tensor mode

Tensor perturbations are invariant under an infinitesimal coordinate transformation. The 6D equation of motion for tensor perturbations is

$$\left(a^4 f h'_{\mu\nu}\right)' + a^2 L_I^2 \Box h_{\mu\nu} = 0, \tag{3.5}$$

where $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$. Setting $\Box = 0$ we obtain the zero-mode solution

$$h_{\mu\nu}(x, w) = \mathtt{A}^I_{\mu\nu}(x) + \mathtt{B}^I_{\mu\nu}(x)\int^w \frac{dw}{a^4 f}, \tag{3.6}$$

where $\mathtt{A}^I_{\mu\nu}$ and $\mathtt{B}^I_{\mu\nu}$ are integration constants. Requiring the regularity of $h_{\mu\nu}$ and $h'_{\mu\nu}$ at the poles [26, 27], we see that $\mathtt{B}^I_{\mu\nu}$ vanishes in the region that contains the pole. Thus, in the single brane model the zero mode is given by $h_{\mu\nu} = \mathtt{A}^I_{\mu\nu}(x)$ everywhere. The source-free Israel condition reads $[[\beta_I h'_{\mu\nu}]]_{\bar{w}_i} = 0$. This means that, also in the two-brane model, the zero mode is written as $h_{\mu\nu} = \mathtt{A}^I_{\mu\nu}(x)$ everywhere. In the next section, we will further discuss the tensor mode with the introduction of matter sources on the brane(s).

B. Scalar modes

In Sec. III B 1 we shall specify the gauges we use. Then in Sec. III B 2 we write down the 6D equations of motion and general solutions in the zero-mode sector, as well as the boundary conditions imposed at the poles and at the branes. Some of the equations will be used in Sec. IV, but one can go directly to the next section without missing essential points.

1. *The gauge choices*

The gauge transformations for the scalar-type variables are summarized in Appendix B. One of the gauge we employ in the present paper is an analogue to the longitudinal gauge, which is defined by $E = B_w = B = 0$. Hence we can define metric perturbations in the longitudinal gauge as

$$\tilde{\Psi} := \Psi - \frac{a'}{a}\frac{f}{L_I^2}X, \tag{3.7}$$

$$\tilde{W} := W - \frac{f}{L^2}X' - \frac{f'}{2L_I^2}X, \tag{3.8}$$

$$\tilde{\Phi} := \Phi - \frac{f'}{2L_I^2}X, \tag{3.9}$$

$$\tilde{C} := C - \left(\frac{B}{\ell^2 f}\right)', \tag{3.10}$$

where we defined a convenient variable $X := B_w - a^2 E'$. Similarly, the gauge field perturbations in the longitudinal gauge can be defined as

$$\tilde{A} := A - \mathcal{A}_\phi \left(\frac{B}{\ell^2 f}\right), \tag{3.11}$$

$$\tilde{A}_w := A_w - \mathcal{A}_\phi \left(\frac{B}{\ell^2 f}\right)', \tag{3.12}$$

$$\tilde{A}_\phi := A_\phi - \mathcal{A}_\phi' \frac{f}{L_I^2} X. \tag{3.13}$$

Hereafter, scalar-type variables with tilde will denote perturbations in the longitudinal gauge.

As will be seen, the 6D perturbation equations in the longitudinal gauge reduce to a system of two (coupled) master equations. For this reason the longitudinal gauge is useful for solving the bulk equations. In this gauge, however, the positions of the branes $w_b^{(i)}$ are perturbed in general:

$$w_b^{(i)} = \bar{w}_i + \zeta^{(i)}(x). \tag{3.14}$$

Strictly speaking, we have two different brane bending modes for each one brane, $\zeta^{(i)}_{w>\bar{w}_i}$ and $\zeta^{(i)}_{w<\bar{w}_i}$, as no Z_2-symmetry is assumed at the branes.

The Gaussian-normal gauge, defined by

$$B_w = W = C = 0 \quad \text{and} \quad w_b^{(i)} = \bar{w}_i, \tag{3.15}$$

is convenient when imposing the boundary conditions at the brane as the brane position is not perturbed. We can employ this gauge in the neighborhood of each brane. Hereafter, scalar-type variables with bar will denote perturbations in the Gaussian-normal gauge. Starting from the longitudinal gauge, the Gaussian-normal gauge is realized by a coordinate transformation $\bar{x}^M = \tilde{x}^M + \xi^M$ such that

$$0 = -a^2 \xi' - \frac{L_I^2}{f} \xi^w, \tag{3.16}$$

$$0 = \tilde{W} - \xi^{w\prime} + \frac{f'}{2f} \xi^w, \tag{3.17}$$

$$0 = \tilde{C} - \xi^{\phi\prime}, \tag{3.18}$$

$$0 = \zeta^{(i)} + \xi^w|_{\bar{w}_i}. \tag{3.19}$$

We can fix the residual gauge freedom by imposing $\xi|_{\bar{w}_i} = \xi^\phi|_{\bar{w}_i} = 0$. With this, perturbative matter sources localized on the brane are invariant when going from the longitudinal gauge to the Gaussian-normal gauge.

The above gauge transformation relates the longitudinal gauge perturbations with the perturbations of the induced metric and gauge field perturbations on the brane. For example, we find

$$\bar{\Psi}|_{\bar{w}_i} = \left[\tilde{\Psi} + \frac{a'}{a}\zeta^{(i)}\right]\Big|_{\bar{w}_i}, \tag{3.20}$$

$$\bar{\Phi}|_{\bar{w}_i} = \left[\tilde{\Phi} + \frac{f'}{2f}\zeta^{(i)}\right]\Big|_{\bar{w}_i}. \tag{3.21}$$

We also have $\bar{A}|_{\bar{w}_i} = \tilde{A}|_{\bar{w}_i}$ and $\bar{F}_w|_{\bar{w}_i} = \tilde{F}_w|_{\bar{w}_i}$.

2. Perturbation equations and boundary conditions

We work in the longitudinal gauge for the analysis of the bulk perturbations. The linearized Einstein equations give

$$\frac{L_I^2}{a^2}\Box\left(\tilde{\Phi}+3\tilde{\Psi}\right)+2f\left[2\frac{a'}{a}\tilde{\Phi}'+\left(\frac{f'}{f}+6\frac{a'}{a}\right)\tilde{\Psi}'\right]+\tilde{W}+\frac{\mu^2}{a^8}\tilde{\Phi}=\frac{\mu}{\ell M^2a^4}\tilde{A}_\phi', \tag{3.22}$$

$$4f\left[\tilde{\Psi}''+\left(\frac{f'}{2f}+5\frac{a'}{a}\right)\tilde{\Psi}'-\frac{a'}{a}\tilde{W}'+\frac{a'}{a}\left(\frac{f'}{f}+3\frac{a'}{a}\right)(\tilde{\Phi}-\tilde{W})\right]$$

$$+\frac{L_I^2}{a^2}\Box\left(3\tilde{\Psi}+\tilde{W}\right)+\tilde{\Phi}+\frac{\mu^2}{a^8}\tilde{W}=\frac{\mu}{\ell M^2a^4}\tilde{A}_\phi', \tag{3.23}$$

$$f\left[\tilde{\Phi}'+3\tilde{\Psi}'+\left(\frac{f'}{2f}-\frac{a'}{a}\right)\tilde{\Phi}-\left(\frac{f'}{2f}+3\frac{a'}{a}\right)\tilde{W}\right]=-\frac{\mu}{\ell M^2a^4}\tilde{A}_\phi, \tag{3.24}$$

$$f\left[\tilde{\Phi}''+3\tilde{\Psi}''-\left(\frac{f'}{2f}+3\frac{a'}{a}\right)\tilde{W}'+3\left(\frac{f'}{2f}+\frac{a'}{a}\right)\tilde{\Phi}'+3\left(\frac{f'}{f}+4\frac{a'}{a}\right)\tilde{\Psi}'\right.$$

$$\left.+\left(\frac{f''}{f}+6\frac{a'}{a}\frac{f'}{f}+6\frac{a'^2}{a^2}\right)(\tilde{\Psi}-\tilde{W})\right]-\frac{\mu^2}{a^8}(\tilde{W}+\tilde{\Phi})+\left(1+\frac{\mu^2}{a^8}\right)\tilde{\Psi}=-\frac{\mu}{\ell M^2a^4}\tilde{A}_\phi', \tag{3.25}$$

$$2\tilde{\Psi}+\tilde{\Phi}+\tilde{W}=0, \tag{3.26}$$

and

$$\Box\tilde{C} = 0, \tag{3.27}$$

$$\tilde{C}'+2\left(\frac{f'}{f}+\frac{a'}{a}\right)\tilde{C} = -\frac{2\mu}{\ell M^2a^4}\tilde{F}_w. \tag{3.28}$$

From the linearized Maxwell equations $\partial_M\delta\left[\sqrt{-g}\mathcal{F}^{MN}\right]=0$ we get

$$\left[a^4\tilde{A}_\phi'+\mu M^2\ell(4\tilde{\Psi}-\tilde{\Phi}-\tilde{W})\right]'+\frac{a^2L_I^2}{f}\Box\tilde{A}_\phi=0, \tag{3.29}$$

and

$$\left(a^2f\tilde{F}_w\right)' = 0, \tag{3.30}$$

$$\Box\tilde{F}_w = 0. \tag{3.31}$$

On each brane we have the perturbed equation of motion for σ_i field: $\partial_{\hat{\mu}}\left[\sqrt{-q}q^{\hat{\mu}\hat{\nu}}\left(\partial_{\hat{\nu}}\sigma_i-\mathrm{e}\mathcal{A}_{\hat{\nu}}\right)\right]=0$, or, explicitly,

$$\partial^\phi\partial_\phi\delta\sigma_i+\frac{1}{a^2}\Box\left(\delta\sigma_i-\mathrm{e}\tilde{A}\right)=0. \tag{3.32}$$

The continuity of the induced metric and the gauge field imposes

$$\left[\left[\tilde{\Psi}+\frac{a'}{a}\zeta\right]\right]_{\bar{w}_i} = 0, \tag{3.33}$$

$$\left[\left[\tilde{\Phi}+\frac{f'}{2f}\zeta\right]\right]_{\bar{w}_i} = 0, \tag{3.34}$$

$$\left[\left[\tilde{A}_\phi+\mathcal{A}_\phi'\zeta\right]\right]_{\bar{w}_i} = 0, \tag{3.35}$$

$$\left[\left[\tilde{A}\right]\right]_{\bar{w}_i} = 0, \tag{3.36}$$

where we simply write $\zeta^{(i)}=\zeta$.

The $(\mu\nu)$ component of the Israel conditions, including the transverse and traceless perturbation, is

$$\left[\left[\frac{L_I}{f^{1/2}}\left(\zeta_{,\mu\nu}-\Box\zeta\eta_{\mu\nu}\right)-\frac{f^{1/2}}{2L_I}a^2h_{\mu\nu}'\right]\right]_{\bar{w}_i}=\frac{T_{\mu\nu}^{(i)}}{M^4}, \tag{3.37}$$

where $T_{\mu\nu}^{(i)}$ is the matter energy-momentum tensor on the brane labeled by i. The four-dimensional trace of Eq. (3.37) reduces to

$$\left[\left[\frac{L_I}{f^{1/2}}\frac{1}{a^2}\Box\zeta\right]\right]_{\bar{w}_i} = -\frac{1}{3}\frac{T_\lambda^{\ \lambda(i)}}{M^4}. \tag{3.38}$$

The $(\phi\phi)$ component of the Israel conditions gives

$$\left[\left[\frac{f^{1/2}}{L_I}\left\{-4\tilde{\Psi}' + 4\frac{a'}{a}\tilde{W} + \left(\frac{f'}{2f} - \frac{a'}{a}\right)\tilde{\Phi} + \frac{L_I^2}{a^2 f}\Box\zeta + \tilde{Y}\right\}\right]\right]_{\bar{w}_i} = -\frac{T_\phi^{\ \phi(i)}}{M^4}, \tag{3.39}$$

where we defined

$$\tilde{Y} := \left(\frac{f'}{2f} - \frac{a'}{a}\right)\left(\frac{f'}{2f} - 4\frac{a'}{a}\right)\zeta + \frac{\mu^2}{a^8 f}\zeta + \frac{\mu}{\ell M^2 a^4 f}\tilde{A}_\phi. \tag{3.40}$$

From the $(\phi\mu)$ component of the Israel conditions we obtain

$$\left[\left[\beta_I \tilde{C}\right]\right]_{\bar{w}_i} + [[\beta_I]]_{\bar{w}_i}\frac{2\mu}{\mathrm{e}\ell M^2 a_i^4 f_i}\left(\delta\sigma_i - \mathrm{e}\tilde{A}\right)\Big|_{\bar{w}_i} = 0, \tag{3.41}$$

where $a_i := a(\bar{w}_i)$ and $f_i := f(\bar{w}_i)$. The ϕ component of the Maxwell junction conditions gives

$$\left[\left[\beta_I\left\{\left(\frac{f'}{2f} - \frac{a'}{a}\right)\left(\tilde{A}_\phi' - \frac{\mu\ell M^2}{a^4}\tilde{W}\right) + \frac{\mu\ell M^2}{a^4}\tilde{Y}\right\}\right]\right]_{\bar{w}_i} = 0, \tag{3.42}$$

while from the μ component we get

$$\left[\left[\beta_I \tilde{F}_w\right]\right]_{\bar{w}_i} = \frac{\ell \mathrm{e} v_i^2}{f_i^{1/2}}\left(\delta\sigma_i - \mathrm{e}\tilde{A}\right)\Big|_{\bar{w}_i}. \tag{3.43}$$

We can eliminate $(\delta\sigma_i - \mathrm{e}\tilde{A})|_{\bar{w}_i}$ from Eqs. (3.41) and (3.43) to get

$$a_i^4 f_i^{1/2}\left[\left[\beta_I\tilde{C}\right]\right]_{\bar{w}_i} + \frac{2\mu}{(\mathrm{e}v_i\ell M)^2}[[\beta_I]]_{\bar{w}_i}\left[\left[\beta_I\tilde{F}_w\right]\right]_{\bar{w}_i} = 0. \tag{3.44}$$

Note here that under an infinitesimal $U(1)$ gauge transformation, $\delta\mathcal{A}_M \to \delta\mathcal{A}_M + \partial_M\Xi$ and $\delta\sigma_i \to \delta\sigma_i + \mathrm{e}\Xi$, where Ξ is independent of ϕ, the above equations of motion and boundary conditions are invariant.

Equations (3.27), (3.28), (3.30), (3.31), and (3.44) supplemented with appropriate boundary conditions at the poles form a closed set of equations that can determine $\tilde{C}$ and $\tilde{F}_w$. Then Eqs. (3.32), (3.36), and (3.41) [or (3.43)] are enough to determine $\delta\sigma_i$ and $\tilde{A}$ on the brane. In Appendix D, we show that $\tilde{C}$ and $\tilde{F}_w$ modes vanish everywhere both for the single brane and two-brane models.

As shown in the case of the codimension two model [26, 27], Eqs. (3.22)–(3.26) and (3.29) can be solved using two master variables defined by

$$\Omega_1 := -\tilde{\Phi} - 3\tilde{\Psi}, \quad \Omega_2 := \tilde{\Psi}. \tag{3.45}$$

With a straightforward computation Eqs. (3.22)–(3.26) and (3.29) can be recast into a set of coupled equations

$$\Omega_1'' + 2\left(\frac{f'}{f} + 5\frac{a'}{a}\right)\Omega_1' - \frac{2}{f}(\Omega_1 + \Omega_2) + \frac{L_I^2}{a^2 f}\Box\Omega_1 = 0, \tag{3.46}$$

$$\Omega_2'' + 4\frac{a'}{a}\Omega_2' + \frac{L_I^2}{2a^2 f}\Box(\Omega_1 + 2\Omega_2) = 0. \tag{3.47}$$

In terms of these master variables, the remaining metric and gauge field perturbations are given by

$$\tilde{W} = \Omega_1 + \Omega_2, \tag{3.48}$$

$$\frac{\mu}{\ell M^2 a^4}\tilde{A}_\phi = f\left[\Omega_1' + \frac{f'}{f}(\Omega_1 + 2\Omega_2) + 2\frac{a'}{a}\Omega_1\right]. \tag{3.49}$$

We are interested in the zero-mode sector because of its importance in describing the long range gravitational force. Setting $\Box = 0$ in Eqs. (3.46) and (3.47), we obtain analytic solutions for the zero modes:

$$\Omega_1 = \frac{1}{5(1-\alpha)^2 f}\left[\frac{p_I(x)}{a^3}+\frac{q_I(x)}{a^6}+u_I(x)a^2+\frac{4v_I(x)}{a}\right], \tag{3.50}$$

$$\Omega_2 = u_I(x)+\frac{v_I(x)}{a^3}, \tag{3.51}$$

where the integration constants p_I, q_I, u_I, v_I are to be determined by the boundary conditions. In the unwarped $\alpha = 1$ case, general solutions for the zero modes are[3]

$$\Omega_1 = \frac{1}{1-w^2}\left[p_I(x)+q_I(x)w+u_I(x)w^2+\frac{1}{3}v_I(x)w^3\right], \tag{3.52}$$

$$\Omega_2 = u_I(x)+v_I(x)w. \tag{3.53}$$

Since $f = 0$ at the poles and Ω_1 should be regular there, we impose

$$f\Omega_1|_{w=\pm1} = 0. \tag{3.54}$$

We also require that $\tilde{A}_\phi = 0$ at the poles, i.e.,

$$\left[(f\Omega_1)' + 2f'\Omega_2\right]_{w=\pm1} = 0. \tag{3.55}$$

A more rigorous argument on the boundary conditions at the poles is found in Ref. [27].

IV. LINEARIZED GRAVITY ON THE BRANE

A. The zero-mode truncation

Following the approach of [23] we shall see the behavior of weak gravity created by matter sources on the brane(s). Rearranging the Israel condition (3.37) we obtain

$$\left[\left[\beta_I a^4 f h'_{\mu\nu}\right]\right]_{\bar{w}_i} = -\mathcal{S}^{(i)}_{\mu\nu}, \tag{4.1}$$

where we collected together the matter sources and the brane bending scalars in the right hand side:

$$\mathcal{S}^{(i)}_{\mu\nu} := 2a_i^2\left\{\frac{\ell f_i^{1/2}}{M^4}\left(T^{(i)}_{\mu\nu}-\frac{T_\lambda^{\ \lambda(i)}}{3}q^{(i)}_{\mu\nu}\right)-\ell^2\left[\left[\beta_I^{-1}\zeta_{,\mu\nu}\right]\right]_{\bar{w}_i}\right\}. \tag{4.2}$$

We can now put the equation of motion (3.5) and the boundary condition (4.1) into a single equation with source terms as

$$\mathcal{O}h_{\mu\nu} = -\sum_i \mathcal{S}^{(i)}_{\mu\nu}\delta(w-\bar{w}_i), \tag{4.3}$$

where we defined an operator

$$\mathcal{O}h_{\mu\nu} := \beta_I\left[a^4 f h'_{\mu\nu}\right]' + a^2\ell^2\beta_I^{-1}\Box h_{\mu\nu}. \tag{4.4}$$

Then using the (retarded) Green function defined by

$$\mathcal{O}G_R(x,w;x',w') = \delta^{(4)}(x-x')\delta(w-w'), \tag{4.5}$$

[3] Eqs. (3.50) and (3.51) do not coincide with Eqs. (3.52) and (3.53) in the limit of $\alpha \to 1$. However, by replacing the coefficients u_I and v_I as $u_I \to u_I + [(5-3\alpha)/3(1-\alpha)]v_I$ and $v_I \to -[2/3(1-\alpha)]v_I$ we obtain an expression for Ω_2 that reduces to Eq. (3.53) in the unwarped limit. We can obtain such an expression for Ω_1 by replacing p_I and q_I in Eq. (3.50) with appropriate combinations of p_I, q_I, u_I, and v_I. The resultant expression is unnecessarily complicated.

we can solve Eq. (4.3):

$$h_{\mu\nu}(x,w) = -\sum_i \int d^4x' \; G_R(x,w;x',\bar{w}_i)\mathcal{S}^{(i)}_{\mu\nu}. \tag{4.6}$$

The Green function is explicitly given by

$$G_R(x,w;x',w') = -\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \sum_n \frac{\psi_n(w)\psi_n(w')}{m_n^2+\mathbf{k}^2-(\omega+i\epsilon)^2}, \tag{4.7}$$

where $\psi_n(w)$ are a complete set of eigenfunctions of

$$[a^4 f\psi_n']' = -a^2 L_I^2 m_n^2 \psi_n \tag{4.8}$$

and are normalized as

$$\ell^2 \int_{-1}^{1} \psi_n(w)\psi_{n'}(w) a^2(w)\beta_I^{-1} dw = \delta_{nn'}. \tag{4.9}$$

Here we are interested, for example, in gravity acting on isolated sources separated by a distance much larger than the bulk length scale ℓ. In such situations, we may expect that the effect of Kaluza-Klein modes is suppressed so that we are justified to truncate the Green function by keeping only the zero-mode contribution. As already shown in the previous section the zero-mode solution is constant, $\psi_0 = \ell_*^{-2}$, where

$$\ell_* := \ell \left[\int_{-1}^{1} \beta_I^{-1} a^2(w) dw\right]^{1/2}. \tag{4.10}$$

Note that we basically have $\ell_* \sim O(\ell)$. The zero-mode truncation of the Green function yields

$$h_{\mu\nu} \approx h^{(m)}_{\mu\nu} + h^{(\zeta)}_{,\mu\nu}, \tag{4.11}$$

where we have separated $h_{\mu\nu}$ into the "matter" and "brane bending" contributions as

$$h^{(m)}_{\mu\nu} \;:=\; -\frac{2\ell}{\ell_*^2 M^4} \sum_i a_i^2 f_i^{1/2} \Box^{-1} \left(T^{(i)}_{\mu\nu} - \frac{T_\lambda^{\;\lambda(i)}}{3} q^{(i)}_{\mu\nu} \right), \tag{4.12}$$

$$h^{(\zeta)} \;:=\; \frac{2\ell^2}{\ell_*^2} \sum_i a_i^2 \Box^{-1} \left[\left[\beta_I^{-1}\zeta\right]\right]_{\bar{w}_i}. \tag{4.13}$$

In order to discuss the behavior of gravity on the brane, we compute the linearized 4D Ricci tensor for the induced metric

$$\bar{q}^{(i)}_{\mu\nu} = a_i^2 \left[\left(1+2\bar{\Psi}\right)\eta_{\mu\nu} + h_{\mu\nu}\right]\Big|_{\bar{w}_i}. \tag{4.14}$$

We can write the Ricci tensor as

$$\bar{R}^{(i)}_{\mu\nu} \;=\; -\frac{1}{2}\Box h^{(m)}_{\mu\nu} + \frac{1}{4}\Box^2 h^{(\zeta)}\eta_{\mu\nu} - \left(\partial_\mu\partial_\nu + \frac{1}{2}\eta_{\mu\nu}\Box\right)\varphi^{(i)}, \tag{4.15}$$

where

$$\varphi^{(i)} \;:=\; \left(2\bar{\Psi} + \frac{1}{2}\Box h^{(\zeta)}\right)\Big|_{\bar{w}_i} = 2\bar{\Psi}|_{\bar{w}_i} + \frac{\ell^2}{\ell_*^2}\sum_j a_j^2 \left[\left[\beta_I^{-1}\zeta\right]\right]_{\bar{w}_j}. \tag{4.16}$$

Using the trace of the Israel condition (3.38), we find that the first two terms give

$$\begin{aligned}
-\frac{1}{2}\Box h^{(m)}_{\mu\nu} + \frac{1}{4}\Box^2 h^{(\zeta)}\eta_{\mu\nu} \;&=\; \sum_i 8\pi G^{(i)} \left[\left(\bar{T}^{(i)}_{\mu\nu} - \frac{\bar{T}_\lambda^{\;\lambda(i)}}{3}\bar{q}^{(i)}_{\mu\nu}\right) - \frac{\bar{T}_\lambda^{\;\lambda(i)}}{6}\bar{q}^{(i)}_{\mu\nu}\right] \\
&=\; \sum_i 8\pi G^{(i)} \left(\bar{T}^{(i)}_{\mu\nu} - \frac{\bar{T}_\lambda^{\;\lambda(i)}}{2}\bar{q}^{(i)}_{\mu\nu}\right),
\end{aligned} \tag{4.17}$$

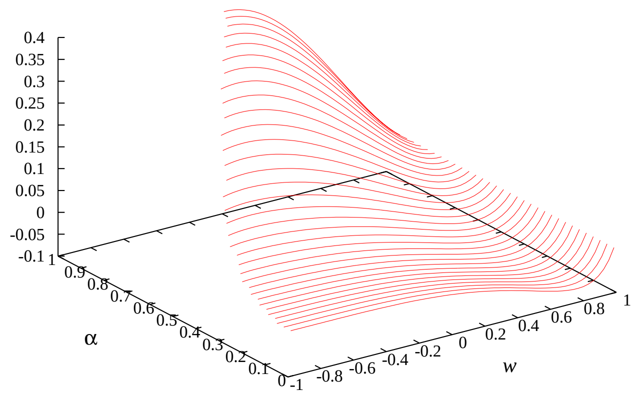

FIG. 3: Plot for $\Delta(\bar{w}, \alpha)$. The location of the brane is restricted in the region $\bar{w} \geq w_c$.

where $\bar{T}_{\mu\nu}$ is the energy-momentum tensor integrated along the ϕ direction,

$$\bar{T}^{(i)}_{\mu\nu} := \int_0^{2\pi} T^{(i)}_{\mu\nu} \sqrt{\bar{q}^{(i)}_{\phi\phi}} d\phi = 2\pi\ell f^{1/2}(\bar{w}_i) T^{(i)}_{\mu\nu}, \tag{4.18}$$

and the 4D gravitational coupling at each brane is defined as

$$8\pi G^{(i)} := \frac{a^2(\bar{w}_i)}{2\pi\ell_*^2 M^4}. \tag{4.19}$$

Now it is clear how the 4D tensor structure can be recovered by the first two terms in Eq. (4.15); the brane bending plays a crucial role. The same mechanism works in the Randall-Sundrum model [23] and in the unwarped, Z_2-symmetric model of [10]. What would make brane gravity different from 4D Einstein is the $\varphi^{(i)}$ term. Apparently, gravity on the brane is described by a scalar-tensor theory due to this scalar mode. However, as we shall show in the next subsection, the effect of $\varphi^{(i)}$ term can be ignored as far as long range gravity is concerned.

B. The $\varphi^{(i)}$ mode

We will now evaluate the impact of $\varphi^{(i)}$ on brane gravity and make clear the physical meaning of this mode. We start with combining Eqs. (3.38) and (3.39) to obtain

$$\left[\left[\beta_I \left\{-4\tilde{\Psi}' + 4\frac{a'}{a}\tilde{W} + \left(\frac{f'}{2f} - \frac{a'}{a}\right)\tilde{\Phi} + \tilde{Y}\right\}\right]\right]_{\bar{w}_i} = \frac{\ell_*^2}{a_i^2} 8\pi G^{(i)} \bar{\tau}^{(i)}, \tag{4.20}$$

where

$$\bar{\tau}^{(i)} := \frac{1}{3}\bar{T}_\lambda^{\ \lambda(i)} - \bar{T}_\phi^{\ \phi(i)}, \tag{4.21}$$

and $\bar{T}_\phi^{\ \phi(i)}$ is defined similarly to $\bar{T}^{(i)}_{\mu\nu}$. This combination of the energy-momentum tensor is the key quantity. To catch the meaning of $\varphi^{(i)}$, the following two quantities are important: the perturbed volume of the internal space $\delta\mathcal{V}$ and the perturbed circumference of each brane $\delta\mathcal{C}^{(i)}$. We can compute the volume of the internal space by performing the integral

$$\mathcal{V}_0 + \delta\mathcal{V} = \int \sqrt{\bar{g}_{ww}\bar{g}_{\phi\phi}} dw d\phi, \tag{4.22}$$

yielding $\mathcal{V}_0 = 2\pi\ell^2 \int \beta_I^{-1} dw$ and

$$\delta\mathcal{V} = -2\pi\ell^2 \left(\sum_i \left[\left[\beta_I^{-1}\zeta\right]\right]_{\bar{w}_i} + 2\int \beta_I^{-1}\Omega_2 dw \right). \tag{4.23}$$

The circumference of the brane is given by

$$\mathcal{C}_0^{(i)} + \delta\mathcal{C}^{(i)} = \int \sqrt{\bar{g}_{\phi\phi}} d\phi, \tag{4.24}$$

and so we have $\mathcal{C}_0^{(i)} = 2\pi\ell f_i^{1/2}$ and

$$\delta\mathcal{C}^{(i)} \;=\; 2\pi\ell f_i^{1/2}\, \bar{\Phi}\big|_{\bar{w}_i}. \tag{4.25}$$

Let us first consider a single brane model. In this case there are $4 \times 2 = 8$ scalar integration constants and 2 brane bending modes. Using 4 boundary conditions at the brane [Eqs. (3.33)–(3.35) and (3.42)] and $2 \times 2 = 4$ regularity conditions at the two poles [Eqs. (3.54) and (3.55)], we can express 8 of 10 variables in terms of 2 variables, say $u_-(x)$ and $v_-(x)$. Then, substituting the result into Eqs. (4.16) and (4.20) we can express φ and $\bar{\tau}$ in terms of $u_-(x)$ and $v_-(x)$. This procedure reveals the relation

$$\varphi = \frac{\ell_*^2}{a^2(\bar{w})} 8\pi G\bar{\tau}\, \Delta(\bar{w},\alpha), \tag{4.26}$$

where $\Delta(\bar{w},\alpha)$ is a regular function and it is dimensionless. Since the explicit form of $\Delta(\bar{w},\alpha)$ is quite messy, we show a plot in Fig. 3 instead of writing down a lengthy equation. Now we have

$$\bar{R}_{\mu\nu} = 8\pi G\left[\left(\bar{T}_{\mu\nu} - \frac{\bar{T}_\lambda^{\ \lambda}}{2}\bar{q}_{\mu\nu}\right) - \Delta(\bar{w},\alpha)\frac{\ell_*^2}{a^2(\bar{w})}\left(\partial_\mu\partial_\nu + \frac{1}{2}\eta_{\mu\nu}\Box\right)\bar{\tau}\right]. \tag{4.27}$$

From this it is easy to see that the second term is negligible on scales much larger than $\ell_*(\sim \ell)$.

The scalar mode φ decouples from gravity on the brane as the brane shrinks to the pole ($\bar{w} \to +1$), since $\Delta \to 0$ in this limit. This is in agreement with the result of [10]. Note, however, that in fact one cannot bring the brane to the pole for a finite value of v because of Eq. (2.21) in the single brane model.

We can also express $\delta\mathcal{V}$ and $\delta\mathcal{C}$ in terms of $u_-(x)$ and $v_-(x)$. With this we find

$$\varphi \propto \delta\mathcal{V} \propto \delta\mathcal{C}. \tag{4.28}$$

This means that the perturbation of the volume of the internal space and that of the circumference of the brane are not independent, and φ corresponds to this mode.

The situation for the two-brane model is similar to that of the above single brane case. There are $4 \times 3 = 12$ integration constants and $2 \times 2 = 4$ brane bending scalars, while we have $4 \times 2 = 8$ boundary conditions at the branes and 4 regularity conditions at the poles. Thus we can express 12 of 16 variables in terms of 4 variables, say $u_0(x)$, $v_0(x)$, $u_-(x)$, and $v_-(x)$. Then Eqs. (4.16) and (4.20) allow us to express $\varphi^{(+)}$, $\bar{\tau}^{(+)}$, and $\bar{\tau}^{(-)}$ in terms of these four variables. Interestingly, with this procedure we can show that all the integration constants in $\varphi^{(+)}$ are encoded solely into $\bar{\tau}^{(\pm)}$ as

$$\varphi^{(+)} = \frac{\ell_*^2}{a_+^2} 8\pi G^{(+)}\left[\bar{\tau}^{(+)}\Delta^{(+)} + \bar{\tau}^{(-)}\Delta^{(-)}\right], \tag{4.29}$$

where $\Delta^{(\pm)}$ are regular functions of $\bar{w}_+, \bar{w}_-, \alpha$, and β_0. Thus we arrive at the same conclusion as the above: the effect of the mode $\varphi^{(+)}$ can be safely neglected on scales much larger than ℓ_* on the brane. Again, the explicit form of $\Delta^{(\pm)}$ is quite involved and so we just illustrate their behavior in Figs. 4–7. It turns out that $\Delta^{(+)} \to 0$ as the brane at $w = \bar{w}_+$ shrinks to the north pole, $\bar{w}_+ \to +1$. Similarly, we have $\Delta^{(-)} \to 0$ as the brane at $w = \bar{w}_-$ shrinks to the south pole, $\bar{w}_- \to -1$. Thus, the $\varphi^{(+)}$ mode decouples from gravity on the brane when one brings both branes to the poles.

It is straightforward to express the perturbations of the circumferences of the two branes, $\delta\mathcal{C}^{(+)}$ and $\delta\mathcal{C}^{(-)}$, and the volume fluctuation $\delta\mathcal{V}$, in terms of $u_0(x)$, $v_0(x)$, $u_-(x)$, and $v_-(x)$. Comparing the result with $\bar{\tau}^{(\pm)}$ expressed in terms of these four variables, we find that $\delta\mathcal{C}^{(\pm)}$ and $\delta\mathcal{V}$ can in fact be written as linear combinations of $\bar{\tau}^{(\pm)}$:

$$\delta\mathcal{C}^{(\pm)} \;=\; c_1^{(\pm)}\bar{\tau}^{(+)} + c_2^{(\pm)}\bar{\tau}^{(-)}, \tag{4.30}$$

$$\delta\mathcal{V} \;=\; d_1\bar{\tau}^{(+)} + d_2\bar{\tau}^{(-)}. \tag{4.31}$$

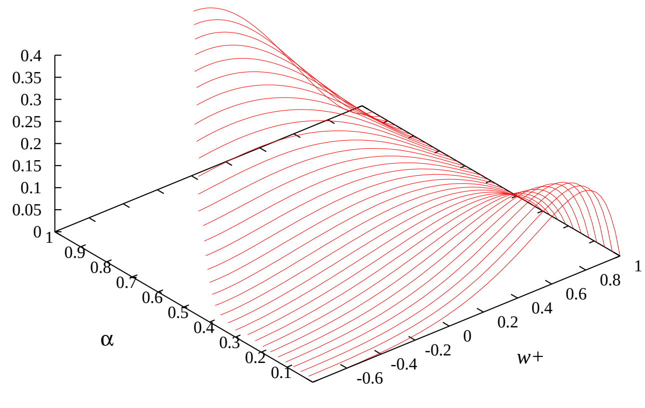

FIG. 4: $\Delta^{(+)}$ as a function of $\bar{w}_+$ and α. The other parameters are fixed as $\bar{w}_- = -0.8$ and $\beta_0 = 0.8$. The domain of $\bar{w}_+$ is restricted from the condition $\bar{w}_+ \geq \max\{\bar{w}_-, w_c\}$.

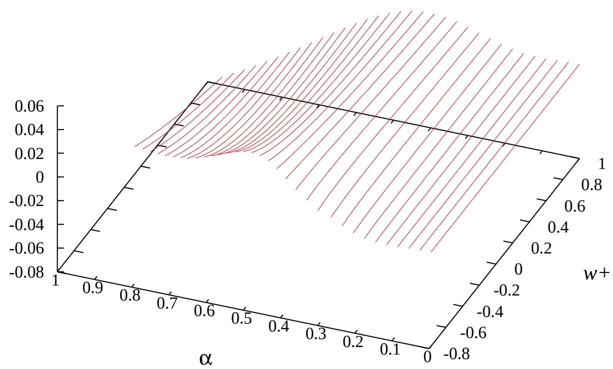

FIG. 5: $\Delta^{(-)}$ as a function of $\bar{w}_+$ and α. The other parameters are fixed as $\bar{w}_- = -0.8$ and $\beta_0 = 0.8$. The domain of $\bar{w}_+$ is restricted from the condition $\bar{w}_+ \geq \max\{\bar{w}_-, w_c\}$.

Namely, two of the three fluctuation modes $\delta\mathcal{C}^{(\pm)}$ and $\delta\mathcal{V}$ are linearly independent. The mode $\varphi^{(+)}$ can be interpreted as a combination of two of these fluctuation modes.

Finally, let us take a brief look at the unwarped $\alpha = 1$ model, focusing on the simplest background configuration with $v_+ = v_-$, i.e., the Z_2-symmetric configuration of the branes ($\bar{w}_- = -\bar{w}_+$). In this case it is instructive to rearrange Eq. (4.29) as

$$\varphi^{(+)} = 8\pi G^{(+)}\ell_*^2 \left[\frac{\Delta^{(+)} + \Delta^{(-)}}{2} \left(\bar{\tau}^{(+)} + \bar{\tau}^{(-)} \right) + \frac{\Delta^{(+)} - \Delta^{(-)}}{2} \left(\bar{\tau}^{(+)} - \bar{\tau}^{(-)} \right) \right]. \tag{4.32}$$

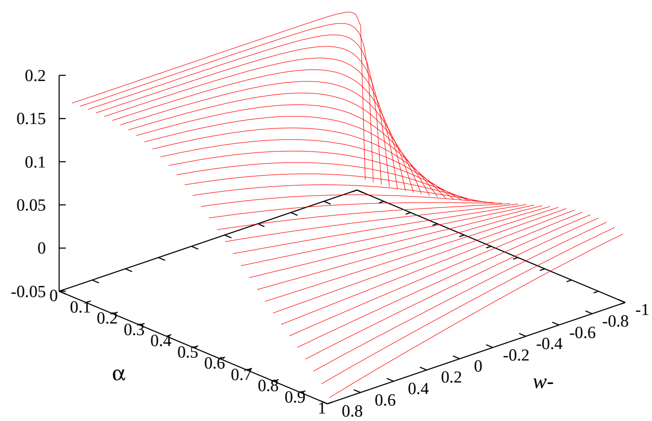

FIG. 6: $\Delta^{(+)}$ as a function of $\bar{w}_-$ and α. The other parameters are fixed as $\bar{w}_+ = 0.8$ and $\beta_0 = 0.6$.

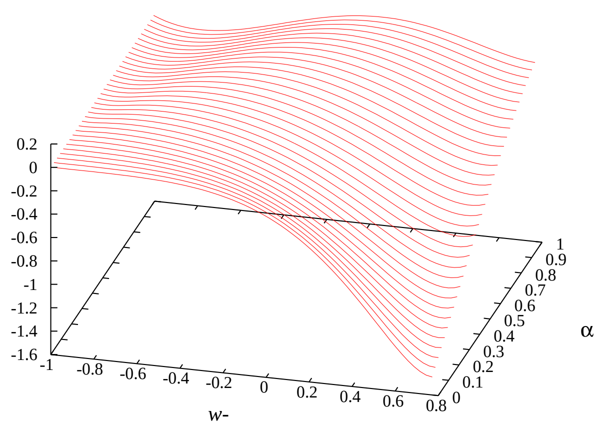

FIG. 7: $\Delta^{(-)}$ as a function of $\bar{w}_-$ and α. The other parameters are fixed as $\bar{w}_+ = 0.8$ and $\beta_0 = 0.6$.

The coefficients are given by

$$\Delta^{(+)} + \Delta^{(-)} = \frac{(1-\bar{w}_+)\left[9\beta + (10-7\beta)\bar{w}_+ - (11+\beta)\bar{w}_+(1+\bar{w}_+)\right]}{12\left[\beta(1-\bar{w}_+) + \bar{w}_+\right]}, \tag{4.33}$$

$$\Delta^{(+)} - \Delta^{(-)} = \frac{\bar{w}_+(1-\bar{w}_+)(10+\bar{w}_+ + \bar{w}_+^2)}{2\left[10\beta + 9(1-\beta)\bar{w}_+ + (1-\beta)\bar{w}_+^3\right]}. \tag{4.34}$$

The restriction to Z_2-symmetric perturbations projects out the second term, and then the result is in agreement with [10]. In other words, Eq. (4.32) generalizes [10] to non-Z_2-symmetric perturbations.

V. CONCLUSIONS

In this paper we have studied linearized gravity on a brane in a 6D Einstein-Maxwell system. In the present braneworld model two extra dimensions are compactified by a magnetic flux, and hence the model is thought of as a toy model of string flux compactifications. Introducing codimension one branes (extended branes) with one spatial dimension compactified on a Kaluza-Klein circle instead of strict codimension two defects, one can put arbitrary energy-momentum tensor on branes [10, 19, 20]. With this we can first discuss the behavior of gravity sourced by matter on the branes. We have considered an axisymmetric, warped background with one or two branes, and examined axisymmetric perturbations in this system, generalizing the previous analysis [10] to the warped model of [20]. We have focused on the zero-mode sector of perturbations as the effect of Kaluza-Klein modes is expected to be subdominant at long distances.

It is a well-known fact that the contribution from the brane bending mode is crucial for recovering the 4D tensor structure in the Randall-Sundrum braneworld [23]. We found that the same mechanism works in the present class of 6D warped braneworlds as well, which has been shown only for the unwarped Z_2-symmetric model in [10]. There appears another scalar mode associated to the fluctuations of the circumferences of the branes, so that gravity is described by a scalar-tensor type theory. However, we have shown that, as in the unwarped case, the scalar mode can be neglected at distances much larger than the scale of flux compactification, $\ell_* \sim \ell$. Therefore, 4D Einstein gravity is reproduced on an extended brane in 6D warped flux compactifications.

On scales about ℓ the effect of Kaluza-Klein modes will not be negligible in addition to the correction from the above scalar mode. The precise evaluation of the corrections from the Kaluza-Klein modes is important for testing the model against gravitational experiments and observations, which is one of our remaining issues. Note, however, that the scale of the circumference of the brane is given by $\ell f^{1/2}(\bar{w}_i)$, which is much smaller than the scale of the internal space ℓ provided that the location of the brane is sufficiently close to a pole. Consequently, the effect of the modes which are inhomogeneous along the ϕ direction will become significant on much smaller scales than ℓ. Note also that even though we have one microscopic Kaluza-Klein direction, the 4D Planck scale is given by Eq. (4.19) [not by $M_{\rm Pl} \sim \ell \cdot \ell f^{1/2}(\bar{w}_i) M^4$]. Thus it is possible to have the scale ℓ of order submillimeter while addressing the hierarchy problem.

Finally we would like to make several remarks on possible extensions of our current work. First, the stability analysis for the present model remains to be done. It would be also interesting to explore cosmology and nonlinear brane gravity on an extend brane, as we are now able to put arbitrary matter on the brane. The cosmic expansion can be described by a moving brane in a warped bulk, with the scale factor mimicked by the warp factor [28]. However, in the present model the warp factor is bounded above and hence it seems difficult to get an ever-expanding brane universe following this line. Note that a de Sitter universe on an extended brane has been constructed straightforwardly for the unwarped bulk [19]. (See also Ref. [6].) Going beyond the Einstein-Maxwell model, we also plan to study aspects of gravity in the supersymmetric version of the present brane model [20, 21, 22].

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APPENDIX A: DERIVATION OF THE BACKGROUND METRIC

We consider the 6D Einstein-Maxwell system described by the action

$$S = \int d^6x\sqrt{-g}\left[\frac{M^4}{2}\left(\mathcal{R} - \frac{1}{L^2}\right) - \frac{1}{4}\mathcal{F}^2\right]. \tag{A1}$$

The field equations derived from this action are

$$\mathcal{R}_{MN} - \frac{1}{2}g_{MN}\mathcal{R} = -\frac{1}{2L^2}g_{MN} + \frac{1}{M^4}\left(\mathcal{F}_{ML}\mathcal{F}_N{}^L - \frac{1}{4}g_{MN}\mathcal{F}^2\right). \tag{A2}$$

We consider a solution obtained by a double Wick rotation from the Reissner-Nordström solution [6, 26, 27]. The metric can be written as

$$\begin{aligned} g_{MN}dx^M dx^N &= \rho^2\eta_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu + \frac{d\rho^2}{\tilde{f}(\rho)} + \tilde{f}(\rho)d\tilde{\phi}^2, \\ \tilde{f}(\rho) &= \frac{D}{\rho^3} - \frac{\rho^2}{20L^2} - \frac{Q^2}{12M^4}\frac{1}{\rho^6}, \end{aligned} \tag{A3}$$

and the field strength is given by

$$\mathcal{F}_{\rho\tilde{\phi}} = \frac{Q}{\rho^4}. \tag{A4}$$

We assume that the metric function $\tilde{f}(\rho)$ has two positive roots ρ_+ and ρ_- and $\tilde{f}(\rho) > 0$ for $\rho_- < \rho < \rho_+$. The integration constants D and Q are conveniently parameterized by $\alpha := \rho_-/\rho_+ (\leq 1)$ as [20]

$$D = \frac{\rho_+^5}{20L^2}\frac{1-\alpha^8}{1-\alpha^3}, \quad Q^2 = \frac{\rho_+^8 M^4}{L^2}\frac{3\alpha^3(1-\alpha^5)}{5(1-\alpha^3)}. \tag{A5}$$

Introducing a new coordinate w defined by

$$a(w) := \frac{1}{2}\left[(1-\alpha)w + 1 + \alpha\right] = \frac{\rho}{\rho_+}, \tag{A6}$$

we can rewrite the metric and the field strength as

$$g_{MN}dx^M dx^N \;=\; a^2\rho_+^2\eta_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu + \frac{L^2 dw^2}{f} + f\cdot\left[\frac{\rho_+}{2L}(1-\alpha)\right]^2 d\tilde{\phi}^2, \tag{A7}$$

$$\mathcal{F}_{\rho\tilde{\phi}} \;=\; \frac{1}{a^4}\frac{M^2}{L}\sqrt{\frac{3\alpha^3(1-\alpha^5)}{5(1-\alpha^3)}}, \tag{A8}$$

where

$$f = \frac{1}{5(1-\alpha)^2}\left[-a^2 + \frac{1-\alpha^8}{1-\alpha^3}\frac{1}{a^3} - \frac{\alpha^3(1-\alpha^5)}{1-\alpha^3}\frac{1}{a^6}\right]. \tag{A9}$$

Note here that f depends only on the parameter α. The rescaling of the coordinates

$$x^\mu = \rho_+\tilde{x}^\mu, \quad \beta\phi \;=\; \frac{\rho_+}{L^2}(1-\alpha)\tilde{\phi}, \tag{A10}$$

leads to the background solution in the main text.

We would like to remark that in the above the Minkowski metric $\eta_{\mu\nu}$ can be replaced by any Ricci flat metric $g^{(4)}_{\mu\nu}$.

APPENDIX B: GAUGE TRANSFORMATIONS FOR SCALAR PERTURBATIONS

Under a scalar gauge transformation

$$\begin{aligned} x^\mu &\to x^\mu + \xi^{,\mu}, \\ w &\to w + \xi^w, \\ \phi &\to \phi + \xi^\phi, \end{aligned} \tag{B1}$$

the metric perturbations transform as

$$\begin{aligned} \Psi &\to \Psi - \frac{a'}{a}\xi^w, \\ E &\to E - \xi, \\ B_w &\to B_w - a^2\xi' - \frac{L_I^2}{f}\xi^w, \\ B &\to B - \ell^2 f\xi^\phi, \\ W &\to W - \xi^{w\prime} + \frac{f'}{2f}\xi^w, \\ \Phi &\to \Phi - \frac{f'}{2f}\xi^w, \\ C &\to C - \xi^{\phi\prime}. \end{aligned} \tag{B2}$$

The perturbed gauge field transforms as

$$\begin{aligned} A &\rightarrow A - \mathcal{A}_\phi \xi^\phi, \\ A_w &\rightarrow A_w - \mathcal{A}_\phi {\xi^\phi}', \\ A_\phi &\rightarrow A_\phi - \xi^w \mathcal{A}'_\phi, \end{aligned} \tag{B3}$$

and so

$$F_w \rightarrow F_w + \mathcal{A}'_\phi \xi^\phi. \tag{B4}$$

APPENDIX C: VECTOR MODES

Under a vector gauge transformation

$$x^\mu \rightarrow x^\mu + \hat{\xi}^\mu, \tag{C1}$$

the metric perturbations transform as

$$\begin{aligned} E_\mu &\rightarrow E_\mu - \hat{\xi}_\mu, \\ B_{w\mu} &\rightarrow B_{w\mu} - a^2 \hat{\xi}'_\mu, \\ B_\mu &\rightarrow B_\mu. \end{aligned} \tag{C2}$$

From Eqs. (C2) we find the two gauge invariant variables

$$V_\mu := B_{w\mu} - a^2 E'_\mu, \tag{C3}$$

and B_μ. The perturbed gauge field $\hat{A}_\mu$ is also gauge invariant.

The linearized Einstein equations give

$$V'_\mu + 2\frac{a'}{a} V_\mu + \frac{f'}{f} V_\mu = 0, \tag{C4}$$

$$\Box V_\mu = 0, \tag{C5}$$

$$B''_\mu + 2\frac{a'}{a} B'_\mu - 6\left(\frac{a'}{a}\right)^2 B_\mu + \frac{L_I^2}{a^2 f} \Box B_\mu = -\frac{\ell}{M^2}\frac{2\mu}{a^4} \hat{A}'_\mu. \tag{C6}$$

From the μ component of the linearized Maxwell equations we obtain

$$\left(a^2 f \hat{A}'_\mu - \frac{M^2}{\ell}\frac{\mu}{a^2} B_\mu \right)' + L_I^2 \Box B_\mu = 0. \tag{C7}$$

The $(\mu\nu)$ component of the Israel conditions gives

$$\frac{a_i^2 f_i^{1/2}}{2\ell} \left[\left[\beta_I \left(V_{\mu,\nu} + V_{\nu,\mu}\right)\right]\right]_{\bar{w}_i} = \frac{1}{M^4} T^{(i)}_{\mu\nu}, \tag{C8}$$

where $a_i := a(\bar{w}_i)$ and $f_i := f(\bar{w}_i)$. From the $(\mu\phi)$ component of the Israel conditions we have

$$\left[\left[\beta_I \left(\frac{1}{2} B'_\mu - \frac{a'}{a} B_\mu \right) \frac{f^{1/2}}{\ell} \right]\right]_{\bar{w}_i} = \mathrm{e} v_i^2 \left(\partial_\phi \sigma - \mathrm{e}\mathcal{A}_\phi\right) \hat{A}_\mu \Big|_{\bar{w}_i}. \tag{C9}$$

The Maxwell junction condition is

$$\left[\left[\beta_I \hat{A}'_\mu \right]\right]_{\bar{w}_i} = \ell (\mathrm{e} v_i)^2 \hat{A}_\mu \Big|_{\bar{w}_i}. \tag{C10}$$

Equations (C6) and (C7) with (C9) and (C10) govern the two vector variables B_μ and $\hat{A}_\mu$. However, they do not couple to matter on the brane via the junction conditions. For this reason we will not take care of these modes.

Equation (C5) indicates that only the zero mode is present for V_μ. Equation (C4) is then solved to give

$$V_\mu(x,w) = \frac{\mathtt{V}^I_\mu(x)}{a^2 f}, \tag{C11}$$

where $\mathtt{V}^I_\mu$ is an integration constant. In order to ensure the regularity at the two poles, we impose $\mathtt{V}^I_\mu = 0$ in the region that contains the pole. In the single brane model, this immediately leads to $V_\mu = 0$ everywhere. In the absence of vector-type matter sources, the source-free Israel condition, Eq. (C8) with $T^{(i)}_{\mu\nu} = 0$, forces V_μ to vanish everywhere in the two-brane model.

APPENDIX D: ON $\tilde{C}$ AND $\tilde{F}_w$

Equation (3.27) shows that only the zero mode is present for $\tilde{C}$. The same is true for $\tilde{F}_w$, as is clear from Eq. (3.31). The general solution to Eqs. (3.28) and (3.30) is given by

$$\tilde{C} = \frac{\mathtt{C}^I(x)}{a^2 f^2} - \mathtt{F}^I(x)\int^w \frac{f}{a^4} dw, \tag{D1}$$

$$\tilde{F}_w = \frac{\ell M^2}{2\mu}\frac{\mathtt{F}^I(x)}{a^2 f}, \tag{D2}$$

where $\mathtt{C}^I$ and $\mathtt{F}^I$ are integration constants. The regularity at the poles requires that $\mathtt{C}^\pm$ and $\mathtt{F}^\pm$ vanish. Thus in the single brane model $\tilde{C} = 0$ and $\tilde{F}_w = 0$ everywhere. In the two-brane model, we can write

$$\tilde{C} = \frac{\mathtt{C}^0(x)}{a^2 f^2} + \mathtt{F}^0(x)\int_w^{\bar{w}_+} \frac{f}{a^4} dw, \tag{D3}$$

$$\tilde{F}_w = \frac{\ell M^2}{2\mu}\frac{\mathtt{F}^0(x)}{a^2 f}, \tag{D4}$$

for $\bar{w}_- < w < \bar{w}_+$. However, the junction condition (3.44) at both branes reads

$$\mathtt{C}^0 + \frac{\beta_+ - \beta_0}{\ell(\mathrm{e}v_+)^2}\frac{f^{1/2}}{a^4}\bigg|_{\bar{w}_+}\mathtt{F}^0 = 0, \tag{D5}$$

$$\mathtt{C}^0 + \left[\left(\int_{\bar{w}_-}^{\bar{w}_+} \frac{f}{a^4} dw\right) a^2 f^2 + \frac{\beta_0 - \beta_-}{\ell(\mathrm{e}v_-)^2}\frac{f^{1/2}}{a^4}\right]\bigg|_{\bar{w}_-}\mathtt{F}^0 = 0, \tag{D6}$$

leading to $\mathtt{C}^0 = 0$ and $\mathtt{F}^0 = 0$. Thus we see that $\tilde{C} = 0$ and $\tilde{F}_w = 0$ everywhere also in the two-brane model. Then from Eq. (3.41) or Eq. (3.43) we have $\delta\sigma_i - \mathrm{e}\tilde{A} = 0$, at the branes. Eq. (3.32) now reduces to $\partial^\phi\partial_\phi\delta\sigma_i = 0$, which is solved by $\delta\sigma_i = \delta\sigma_i^{(1)}(x) + \delta\sigma_i^{(2)}(x)\phi$. However, $\delta\sigma_i^{(2)} = 0$ because of periodicity in ϕ of σ_i and the fact that $\delta\sigma_i$ is infinitesimal. Thus we get $\delta\sigma_i = \mathrm{e}\tilde{A}|_{\bar{w}_i} = \delta\sigma_i^{(1)}(x)$.

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